

Diagonalizing "compact" operators on Hilbert W^* -modules

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Abstract. For W^* -algebras A and self-dual Hilbert A -modules \mathcal{M} we show that every self-adjoint, "compact" module operator on \mathcal{M} is diagonalizable. Some specific properties of the eigenvalues and of the eigenvectors are described.

Keywords: *diagonalization of "compact" operators, Hilbert W^* -modules, W^* -algebras, eigenvalues, eigenvectors*

AMS subject classification: Primary 47C15, secondary 46L99, 46H25, 47A75.

The goal of the present short note is to consider self-adjoint, "compact" module operators on self-dual Hilbert W^* -modules (which can be supposed to possess a countably generated W^* -predual Hilbert W^* -module, in general) with respect to their diagonalizability. Some special properties of their eigenvalues and eigenvectors are described. A partial result in this direction was recently obtained by V. M. Manuilov [10,11] who proved that every such operator on the standard countably generated Hilbert W^* -module $l_2(A)$ over *finite* W^* -algebras A can be diagonalized on the respective A -dual Hilbert A -module $l_2(A)'$. The same was shown to be true for every self-adjoint bounded module operator on finitely generated Hilbert C^* -modules over general W^* -algebras by R. V. Kadison [5,6,7] and over commutative AW^* -algebras by K. Grove and G. K. Pedersen [4] sometimes earlier. M. Frank has made an attempt to find a generalized version of the Weyl-Berg theorem in the $l_2(A)'$ setting for some (abelian) monotone complete C^* -algebras which should satisfy an additional condition, as well as a counterexample, cf. [2]. Further results on generalizations of the Weyl-von Neumann-Berg theorem can be found e. g. in papers of G. J. Murphy [12], S. Zhang [15,16] and H. Lin [9].

We go on to investigate situations where non-finite W^* -algebras appear as coefficients of the special Hilbert W^* -modules under consideration (Proposition 5), and where arbitrary self-dual Hilbert W^* -modules are considered (Theorem 9). The applied techniques are rather different from that in [10,11]. By the way, the results of V. M. Manuilov in [10,11] are obtained to be valid for arbitrary self-adjoint, "compact" module operators on the self-dual Hilbert A -module $l_2(A)'$ over finite W^* -algebras (Proposition 3). This generalizes [10] since in the situation of finite W^* -algebras A the set of "compact" operators on $l_2(A)$ may be definitely smaller than that on $l_2(A)'$, and the latter may not contain all bounded module operators on $l_2(A)'$, in general. We characterize the role of self-duality for getting adequate results in the finite W^* -case (Proposition 4). The final

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result of our investigations is Theorem 9 describing the diagonalizability of "compact" operators on self-dual Hilbert W^* -modules in a great generality.

We consider Hilbert W^* -modules $\{\mathcal{M}, \langle \cdot, \cdot \rangle\}$ over general W^* -algebras A , i. e. (left) A -modules \mathcal{M} together with an A -valued inner product $\langle \cdot, \cdot \rangle : \mathcal{M} \times \mathcal{M} \rightarrow A$ satisfying the conditions:

- (i) $\langle x, x \rangle \geq 0$ for every $x \in \mathcal{M}$.
- (ii) $\langle x, x \rangle = 0$ if and only if $x = 0$.
- (iii) $\langle x, y \rangle = \langle y, x \rangle^*$ for every $x, y \in \mathcal{M}$.
- (iv) $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$ for every $a, b \in A, x, y, z \in \mathcal{M}$.
- (v) \mathcal{M} is complete with respect to the norm $\|x\| = \|\langle x, x \rangle\|_A^{1/2}$.

We always suppose, that the linear structures of the W^* -algebra A and of the (left) A -module \mathcal{M} are compatible, i. e. $\lambda(ax) = (\lambda a)x = a(\lambda x)$ for every $\lambda \in \mathbf{C}, a \in A, x \in \mathcal{M}$. Let us denote the A -dual Banach A -module of a Hilbert A -module $\{\mathcal{M}, \langle \cdot, \cdot \rangle\}$ by $\mathcal{M}' = \{r : \mathcal{M} \rightarrow A : r \text{ is } A\text{-linear and bounded}\}$.

Hilbert W^* -modules have some very nice properties in contrast to general Hilbert C^* -modules: First of all, the A -valued inner product can always be lifted to an A -valued inner product on the A -dual Hilbert A -module \mathcal{M}' via the canonical embedding of \mathcal{M} into \mathcal{M}' , $x \rightarrow \langle \cdot, x \rangle$, turning \mathcal{M}' into a (left) self-dual Hilbert A -module, $(\mathcal{M}' = (\mathcal{M}')')$. Moreover, one has the following criterion on self-duality:

Proposition 1. [1, Thm. 3.2] *Let A be a W^* -algebra and $\{\mathcal{M}, \langle \cdot, \cdot \rangle\}$ be a Hilbert A -module. Then the following conditions are equivalent:*

- (i) \mathcal{M} is self-dual.
- (ii) *The unit ball of \mathcal{M} is complete with respect to the topology τ_1 induced by the semi-norms $\{f(\langle \cdot, \cdot \rangle)^{1/2}\}$ on \mathcal{M} , where f runs over the normal states of A .*
- (iii) *The unit ball of \mathcal{M} is complete with respect to the topology τ_2 induced by the linear functionals $\{f(\langle \cdot, x \rangle)\}$ on \mathcal{M} where f runs over the normal states of A and x runs over \mathcal{M} .*

Furthermore, on self-dual Hilbert W^* -modules every bounded module operator has an adjoint, and the Banach algebra of all bounded module operators is actually a W^* -algebra. And last but not least, every bounded module operator on a Hilbert W^* -module $\{\mathcal{M}, \langle \cdot, \cdot \rangle\}$ can be continued to a unique bounded module operator on its A -dual Hilbert W^* -module \mathcal{M}' preserving the operator norm. (Cf. [13].)

We want to consider (self-adjoint,) "compact" module operators on Hilbert W^* -modules. By G. G. Kasparov [8] an A -linear bounded module operator K on a Hilbert A -module $\{\mathcal{M}, \langle \cdot, \cdot \rangle\}$ is "compact" if it belongs to the norm-closed linear hull of the elementary operators

$$\{\theta_{x,y} : \theta_{x,y}(z) = \langle z, x \rangle y, x, y \in \mathcal{M}\}$$

The set of all "compact" operators on \mathcal{M} is denoted by $K_A(\mathcal{M})$. By [13, Thm. 15.4.2] the C^* -algebra $K_A(\mathcal{M})$ is a two-sided ideal of the set of all bounded, adjointable module

operators $End_A^*(\mathcal{M})$ on \mathcal{M} , and both these sets coincide if and only if \mathcal{M} is algebraically finitely generated as an A -module, (cf. also [3, Appendix]). This will be used below. Since we are going to investigate single "compact" operators we make the useful observation that both the range of a given "compact" operator and the support of it are Hilbert C^* -modules generated by countably many elements with respect to the norm topology or at least with respect to the τ_1 -topology, (cf. Proposition 1). Hence, without loss of generality we can restrict our attention to countably generated Hilbert W^* -modules and their W^* -dual Hilbert W^* -modules.

We are especially interested in the Hilbert W^* -module

$$l_2(A) = \{ \{a_i : i \in \mathbf{N}\} : a_i \in A, \sum_i a_i a_i^* \text{ converges with respect to } \|\cdot\|_A \}$$

$$\langle \{a_i\}, \{b_i\} \rangle = \|\cdot\|_A - \lim_{N \in \mathbf{N}} \sum_{i=1}^N a_i b_i^* ,$$

and in its A -dual Hilbert W^* -module

$$l_2(A)' = \left\{ \{a_i : i \in \mathbf{N}\} : a_i \in A, \sup_{N \in \mathbf{N}} \left\| \sum_{i=1}^N a_i a_i^* \right\| < \infty \right\}$$

$$\langle \{a_i\}, \{b_i\} \rangle = w^* - \lim_{N \in \mathbf{N}} \sum_{i=1}^N a_i b_i^* ,$$

because of G. G. Kasparov's stabilization theorem [8], stating that every countably generated Hilbert C^* -module over a unital C^* -algebra A is a direct summand of $l_2(A)$.

Definition 2. Let A be a W^* -algebra and let $\{\mathcal{M}, \langle \cdot, \cdot \rangle\}$ be a self-dual Hilbert A -module possessing a countably generated Hilbert A -module as its A -predual. A bounded module operator T on \mathcal{M} is *diagonalizable* if there exists a sequence $\{x_i : i \in \mathbf{N}\}$ of non-trivial elements of \mathcal{M} such that:

- (i) $T(x_i) = \Lambda_i x_i$ for some elements $\Lambda_i \in A$, ($i \in \mathbf{N}$),
- (ii) The Hilbert A -submodule generated by the elements $\{x_i\}$ inside \mathcal{M} has a trivial orthogonal complement.
- (iii) The elements $\{x_i : i \in \mathbf{N}\}$ are pairwise orthogonal, and the values $\{p_i = \langle x_i, x_i \rangle : i \in \mathbf{N}\}$ are projections in A .
- (iv) The equality $\Lambda_i p_i = \Lambda_i$ holds for the projection p_i , ($i \in \mathbf{N}$).

Note, that the eigenvalues and the eigenvectors are not uniquely determined for the operator T since $T(x) = \Lambda x$ implies $T(y) = \Lambda' y$ for $\Lambda' = u \Lambda u^*$ and $y = ux$ for all unitaries $u \in A$. Moreover, the eigenvalues of T do not belong to the center of A , in general. Consequently, $T(ax) = a(\Lambda x) \neq \Lambda(ax)$, in general. That is, eigenvectors are often not one-to-one related to T -invariant A -submodules of the Hilbert A -module \mathcal{M} under consideration.

Now, we start our investigations decomposing A into components of prescribed type with respect to its direct integral representation. Denote by p that central projection of A dividing A into a finite part pA and into an infinite part $(1-p)A$. That means, that with respect to the direct integral decomposition of A the fibers are almost everywhere factors of type I_n , $n < \infty$, or II_1 inside pA and almost everywhere factors of type I_∞ or II_∞ or III inside $(1-p)A$. Analogously, the Hilbert A -module $l_2(A)$ decomposes into the direct sum of two Hilbert A -modules $l_2(A) = l_2(pA) \oplus l_2((1-p)A)$, and every bounded A -linear operator T on $l_2(A)$ splits into the direct sum $T = pT \oplus (1-p)T$,

where each part acts only on the respective part of the Hilbert A -module non-trivially and at the same time as an A -linear operator.

Consequently, we can proceed considering W^* -algebras A of coefficients of prescribed type. Our first goal is to revise the case of finite W^* -algebras investigated by V. M. Manuilov. There the set $K_A(l_2(A)')$ does not coincide with the set $End_A(l_2(A)')$, and there are always self-adjoint, bounded module operators T on $l_2(A)'$ which can not be diagonalized. For example, consider a self-adjoint, bounded linear operator T_o on a separable Hilbert space H being non-diagonalizable, (cf. Weyl's theorem). Using the decomposition $l_2(A) = \overline{A \otimes H}$ one obtains a self-adjoint, bounded module operator T on $l_2(A)$ by the formula $T(a \otimes h) = a \otimes T_o(h)$, ($a \in A$, $h \in H$). The operator T extends to an operator on $l_2(A)'$, and T can not be diagonalizable by assumption. Surprisingly, V. M. Manuilov proved that every self-adjoint, "compact" operator on the standard countably generated Hilbert W^* -module $l_2(A)$ over finite W^* -algebras A can be diagonalized on the respective A -dual Hilbert A -module $l_2(A)'$. A careful study of his detailed proofs at [10], [11] brings to light that for finite W^* -algebras with infinite center the continuation of the "compact" operators to the respective A -dual Hilbert A -module is not only a proof-technical necessity, but it is of principal character. Self-duality has to be supposed to warrant the diagonalizability of all self-adjoint "compact" module operators on $\mathcal{M} \subseteq l_2(A)'$ in the finite case, and the key steps of the proof can be repeated one-to-one. Consequently, we give the generalized formulation of V. M. Manuilov's diagonalization theorem for the finite case, and we show additionally that self-duality is an essential property of Hilbert W^* -modules for finding a (well-behaved) diagonalization of arbitrary "compact" module operators on them, in general.

Proposition 3. (cf. [10], [11, Thm.4.1]) *Let A be a W^* -algebra of finite type. Then every self-adjoint, "compact" module operator K on $l_2(A)'$ is diagonalizable. The sequence of eigenvalues $\{\Lambda_n : n \in \mathbf{N}\}$ of K has the property $\lim_{n \rightarrow \infty} \|\Lambda_n\| = 0$. The eigenvalues $\{\Lambda_n : n \in \mathbf{N}\}$ of K can be chosen in such a way that $\Lambda_2 \leq \Lambda_4 \leq \dots \leq 0 \leq \dots \leq \Lambda_3 \leq \Lambda_1$. Moreover, for positive operators K without kernel the eigenvectors $\{x_n : n \in \mathbf{N}\}$ may possess the property $\langle x_n, x_n \rangle = 1_A$, ($n \in \mathbf{N}$), in addition.*

For the detailed (but extended) proof of this proposition see [11], (also [10]). The proving technique relies mainly on spectral decomposition theory of operators and on the center-valued trace on the finite W^* -algebra A .

Proposition 4. *Let A be a finite W^* -algebra with infinite center. Consider a Hilbert A -module \mathcal{M} such that $l_2(A) \subset \mathcal{M} \subseteq l_2(A)'$. Then the following two statements are equivalent:*

- (i) $\mathcal{M} = l_2(A)'$, i.e., \mathcal{M} is self-dual.
- (ii) Every positive "compact" module operator is diagonalizable inside \mathcal{M} with comparable inside the positive cone of A eigenvalues.

Proof. Note, that $l_2(A) \neq l_2(A)'$ by assumption. Denote the standard orthonormal basis of $l_2(A)$ by $\{e_n : n \in \mathbf{N}\}$. If the center of A is supposed to be infinite dimensional then one finds a sequence of pairwise orthogonal non-trivial projections $\{p_n : n \in \mathbf{N}\} \in Z(A)$ summing up to 1_A in the sense of w^* -convergence. Fix a sequence of positive non-

zero numbers $\{\alpha_n : n \in \mathbf{N}\}$ monotonically converging to zero. The bounded module operator K defined by

$$K(e_1) = (\sum_{n=1}^{\infty} \alpha_n p_n e_n), \quad K(e_j) = \alpha_j p_j e_1 \text{ for } j \neq 1$$

is a "compact" operator on $l_2(A)$. It easily continues to a "compact" operator on \mathcal{M} . As an exercise one checks that the eigenvalues of K are $\{\alpha_1 p_1, \alpha_2 p_2, \dots, 0, \dots, -\alpha_2 p_2\}$ (ordering by sign and norm and taking into account (iii) and (iv) of Definition 2), and that the appropriate eigenvectors are

$$\{p_1 e_1, 1/\sqrt{2} p_2 (e_1 + e_2), 1/\sqrt{2} p_3 (e_1 + e_3), \dots, \{(1_A - p_n) e_n : n \in \mathbf{N}\}, \dots, 1/\sqrt{2} p_3 (e_1 - e_3), 1/\sqrt{2} p_2 (e_1 - e_2)\}.$$

The only way of making the eigenvalues comparable inside the positive cone of A preserving Definition 2, (iii)-(iv) is to sum up the positive and the negative eigenvalues separately. But, then the resulting eigenvector

$$x = (1_A + (1 + 1/\sqrt{2})(1_A - p_1), 1/\sqrt{2} p_2, 1/\sqrt{2} p_3, \dots, 1/\sqrt{2} p_n, \dots),$$

corresponding to the only positive eigenvalue $\sum_{n=1}^{\infty} \alpha_n p_n$ of K does not belong to \mathcal{M} any longer by assumption. This shows one implication. The converse implication follows from Proposition 6. ■

The second big step is to investigate the case of infinite W^* -algebras as coefficients of the Hilbert W^* -modules under consideration. The result is characteristic for the situation in self-dual Hilbert W^* -modules over infinite W^* -algebras, and quite different from that in the finite W^* -case, and elsewhere, from the classical Hilbert space situation.

Proposition 5. *Let A be a W^* -algebra which possesses infinitely many pairwise orthogonal, non-trivial projections $\{p_i : i \in \mathbf{N}\}$ equivalent to 1_A and summing up to 1_A in the sense of w^* -convergence of the sum $\sum_i p_i = 1_A$. Then the Hilbert A -module $l_2(A)'$ equipped with its standard A -valued inner product is isomorphic to the Hilbert A -module $\{A, \langle \cdot, \cdot \rangle_A\}$, where $\langle a, b \rangle_A = ab^*$.*

Proof. Suppose, the equivalence of the projections $\{p_i : i \in \mathbf{N}\}$ with 1_A is realized by partial isometries $\{u_i : i \in \mathbf{N}\} \in A$, $p_i = u_i u_i^*$, $1_A = u_i^* u_i$. Then the mapping

$$S : l_2(A)' \rightarrow A, \quad \{a_i\} \rightarrow w^* - \lim(\text{finite } \sum_i a_i u_i^*)$$

with the inverse mapping

$$S^{-1} : A \rightarrow l_2(A)', \quad a \rightarrow \{a u_i\}$$

realizes the isomorphism of $l_2(A)'$ and of A as Hilbert A -modules because of Proposition 1. ■

Corollary 6. *Let A be a W^* -algebra of infinite type. Then every bounded module operator T on $l_2(A)'$ is diagonalizable, and the formula*

$$T(\{a_i\}) = \langle \{a_i\}, \{u_i\} \rangle \Lambda_T \{u_i\}$$

holds for every $\{a_i\} \in l_2(A)'$, some $\Lambda_T \in A$ and the partial isometries $\{u_i\} \in A$ described in the previous proof.

Proof. Every W^* -algebra of type I_∞ , II_∞ or III possesses a set of partial isometries with properties described at Proposition 3. The same is true for W^* -algebras consisting only of parts of these types. Now, translate the operator T on $l_2(A)'$ to an operator

STS^{-1} on A and vice versa using Proposition 3, and take into account that every bounded module operator on A is a multiplication operator with a concrete element (from the right). ■

Corollary 7. *Let A be a W^* -algebra without any fibers of type I_n , $n < \infty$, and II_1 in its direct integral decomposition. Let \mathcal{M} be a self-dual Hilbert A -module possessing a countably generated A -predual Hilbert A -module. Then every bounded module operator T on \mathcal{M} is diagonalizable, and the formula*

$$T(x) = \langle x, u \rangle \Lambda_T u$$

holds for every $x \in \mathcal{M}$, some $\Lambda_T \in A$ and a universal for all T eigenvector $u \in \mathcal{M}$.

Proof. Since \mathcal{M} has a countably Hilbert A -module as its A -predual, \mathcal{M} is a direct summand of the Hilbert A -module $l_2(A)'$ by G. G. Kasparov's stabilization theorem ([8]). Hence, one has to show the assertion for the self-dual Hilbert A -module $l_2(A)'$ only. For further use denote the projection from $l_2(A)'$ onto \mathcal{M} by P . Consider the direct integral decomposition of A over its center. Therein every fiber is a W^* -factor of type I_∞ , II_∞ or III by assumption. Putting it into the $l_2(A)'$ -context one obtains that A is isomorphic to $l_2(A)'$ either applying Corollary 6 fiberwise or constructing a suitable set of partial isometries $\{u_i\} \in A$ to make use of Proposition 5. Then in the same way as there the diagonalization result turns out for arbitrary bounded module operators T on $l_2(A)'$. To get the formula of Corollary 7 one has only to set $u = P(\{u_i\})$. ■

Remark. Let A be a I_∞ -factor, for example. Then there are self-adjoint elements Λ_T in A which can not be diagonalized in a stronger sense. More precisely, there is no way of representing any such operator as a sum $\sum \lambda_i P_i$ with $\lambda_i \in \mathbf{C} = Z(A)$ and $P_i = P_i^* = P_i^2 \in A$ because of Weyl's theorem. Therefore, the Corollaries 6 and 7 are the strongest results one could expect.

Example 8. Consider the C^* -algebra A of all 2×2 -matrices on the complex numbers. Set $\mathcal{M} = A^2$ with the usual A -valued inner product. Consider the ("compact") bounded module operator $K = \theta_{x,x} + \theta_{y,y}$ for

$$x = \left(\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right), y = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right).$$

Eigenvectors of K are $x, y \in A^2$, for example, and the respective eigenvalues are

$$\Lambda_x = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}, \Lambda_y = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}.$$

Remark, that one can not compare these eigenvalues as elements of the positive cone of A . But, making another choice one arrives at that situation described at Proposition 6:

$$x_1 = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right), x_2 = \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right).$$

Then the respective eigenvalues are

$$\Lambda_1 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, \Lambda_2 = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}.$$

and they can be ordered as well as the eigenvectors x_1, x_2 are units. Last but not least, dropping out condition (iv) of Definition 2 one can correlate K -invariant submodules of \mathcal{M} and eigenvectors of K . Simply, set

$$x_1 = \left(\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right), x_2 = \left(\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right).$$

In this case the corresponding eigenvalues are

$$\Lambda_1 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, \Lambda_2 = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}.$$

They can be ordered in the positive cone of A . But, the eigenvectors corresponding to the K -invariant submodules of \mathcal{M} can not be selected to be units any longer.

Theorem 9. *Let A be a W^* -algebra and \mathcal{M} be a self-dual Hilbert A -module. Then every self-adjoint, "compact" module operator on \mathcal{M} is diagonalizable. The sequence of eigenvalues $\{\Lambda_n : n \in \mathbf{N}\}$ of K has the property $\lim_{n \rightarrow \infty} \|\Lambda_n\| = 0$. The eigenvalues $\{\Lambda_n : n \in \mathbf{N}\}$ of K can be chosen in such a way that $\Lambda_2 \leq \Lambda_4 \leq \dots \leq 0 \leq \dots \leq \Lambda_3 \leq \Lambda_1$, and that $\{\Lambda_n : n \geq 3\}$ are contained in the finite part of \mathbf{A} .*

Proof. Both the τ_1 -closure of the range and of the support of K are self-dual Hilbert C^* -modules possessing countably generated A -predual Hilbert A -modules because of the "compact"ness of K . Hence, without loss of generality one can restrict the attention to self-dual Hilbert W^* -modules with countably generated W^* -predual Hilbert W^* -modules formed as the τ_1 -completed direct sum of range and support of K . As usual, on the kernel of K one has the eigenvalue zero and a suitable system of eigenvectors. Now, gluing Corollary 4 and Proposition 6 together the theorem turns out to be true in the special case $\mathcal{M} = l_2(A)'$, (cf. the remarks in the beginning of the present note). The only loss may be that the eigenvectors are not units, in general. Because of G. G. Kasparov's stabilization theorem ([8]) \mathcal{M} possesses an embedding into $l_2(A)'$ as a direct summand by assumption. Therefore, every self-adjoint, "compact" module operator K on \mathcal{M} can be continued to a unique such operator on $l_2(A)'$ preserving the norm, simply applying the rule $K|_{\mathcal{M}^\perp} = 0$. The eigenvectors of this extension are elements of \mathcal{M} . The Hilbert A -module \mathcal{M}^\perp belongs to its kernel. This shows the theorem. ■

Remark. For commutative AW^* -algebras A the statement of Theorem 9 is still true by [4]. The general AW^* -case is open at present because of two crucial unsolved problems in the AW^* -theory: (i) Are the self-adjoint elements of $M_n(A)$, $n \geq 2$, diagonalizable for arbitrary (monotone complete) AW^* -algebras A , or not? (ii) Does every finite (monotone complete) AW^* -algebra possess a center-valued trace, or not?

Remark. One can extend the statement of Theorem 9 to the case of normal, "compact" module operators dropping out only the ordering of the eigenvalues. To see this note that for normal elements K of the C^* -algebra $K_A(\mathcal{M})$ there always exists a self-adjoint element $K' \in K_A(\mathcal{M})$ such that K is contained in that C^* -subalgebra of $End_A(\mathcal{M})$ generated by K' and by the identity operator. Applying functional calculus inside the W^* -algebra $End_A(\mathcal{M})$ the result turns out. Beside this, it would be interesting to investigate some more general variants of the Weyl-von Neumann-Berg theorem for appropriate bounded module operators on (self-dual) Hilbert W^* -submodules over (finite) W^* -algebras A as those obtained by H. Lin, G. J. Murphy and S. Zhang.

Acknowledgement. The second author thanks for the partial support by the Russian Foundation for Fundamental Research (grant no. 94-01-00108a) and by the International Science Foundation (grant no. MGM 000). The research work was carried out during a stay at Leipzig which was part of a university cooperation project financed by Deutscher Akademischer Austauschdienst. We are very appreciated to the referees for their remarks on the first version of the present work.

References

- [1] Frank, M.: *Self-duality and C^* -reflexivity of Hilbert C^* -modules*. Zeitschr. Anal. Anwendungen 9(1990), 165-176.
- [2] Frank, M.: *Hilbert C^* -modules over monotone complete C^* -algebras and a Weyl-Berg type theorem*. preprint 3/91, Universität Leipzig, NTZ, 1991. To appear in Math. Nachrichten.
- [3] Frank, M.: *Geometrical aspects of Hilbert C^* -modules*. preprint 22/93, Københavns Universitet, Matematisk Institut, 1993.
- [4] Grove, K. and G. K. Pedersen: *Diagonalizing matrices over $C(X)$* . J. Functional Analysis 59(1984), 64-89.
- [5] Kadison, R. V.: *Diagonalizing matrices over operator algebras*. Bull. Amer. Math. Soc. 8(1983), 84-86.
- [6] Kadison, R. V.: *Diagonalizing matrices*. Amer. J. Math. 106(1984), 1451-1468.
- [7] Kadison, R. V.: *The Weyl theorem and block decompositions*. In: Operator Algebras and Applications, v. 1, Cambridge: Cambridge University Press 1988, pp. 109-117.
- [8] Kasparov, G. G.: *Hilbert C^* -modules: The theorems of Stinespring and Voiculescu*. J. Operator Theory 4(1980), 133-150.
- [9] Lin, H.: *The generalized Weyl - von Neumann theorem and C^* -algebra extensions*. In: Algebraic Methods in Operator Theory. (eds.: R. Curto and P. E. T. Jørgensen), Birkhäuser, Boston - Basel - Berlin, 1994.
- [10] Manuilov, V. M.: *Diagonalization of compact operators on Hilbert modules over W^* -algebras of finite type (in russ.)* Uspekhi Mat. Nauk 49(1994), no. 2, 159-160.
- [11] Manuilov, V. M.: *Diagonalization of compact operators on Hilbert modules over W^* -algebras of finite type (in engl.)* submitted to Annals Global Anal. Geom., 1994.
- [12] Murphy, G. J.: *Diagonality in C^* -algebras*. Math. Zeitschr. 199(1988), 199-229.
- [13] Paschke, W. L.: *Inner product modules over B^* -algebras*. Trans. Amer. Math. Soc. 182(1973), 443- 468.
- [14] Wegge-Olsen, N. E.: *K -theory and C^* -algebras - a friendly approach*. Oxford University Press, Oxford-New York-Tokyo, 1993.
- [15] Zhang, S.: *Diagonalizing projections in the multiplier algebras and matrices over a C^* -algebra*. Pacific J. Math. 145(1990), 181-200.
- [16] Zhang, S.: *K_1 -groups, quasidiagonality and interpolation by multiplier projections*. Trans. Amer. Math. Soc. 325(1991), 793-818.

To appear in *Zeitschrift für Analysis und ihre Anwendungen (ZAA)* v. 14(1995), no. 1, 33-41.